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# Coherent states and dissipation for the motion of a charged particle in a constant magnetic field

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## Abstract

We deal with the motion of a charged particle in a constant magnetic field when immersed in a medium that exerts some type of friction. We analyse the problem classically and also quantum mechanically. In the latter case we use coherent state solutions of the problem that, for large energies, compared with the one associated with the cyclotron frequency, reduce to the classical limit. To study the dynamical behaviour we use time-dependent wave packets that can be constructed by superposition of the coherent states.

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## 1. Introduction

In recent publications we have been interested in understanding dissipation effects in simple systems, both classically and quantum mechanically [1]. There are different ways of taking the dissipative interaction with an environment into account. The traditional system-plus-reservoir approach couples the relevant system to a large number of environmental degrees of freedom, represented, e.g., by harmonic oscillators [2]. However, this has the disadvantage—from a computational point of view—that a system of many coupled differential equations has to be solved due to the large number of degrees of freedom of the reservoir. The number of degrees of freedom can be drastically reduced if an effective description of the interaction between system and reservoir is applied. On the quantum-mechanical level, the effect of the environment can be included, e.g., by the use of explicitly time-dependent Hamiltonians [3] or nonlinear additions to the Schrödinger equation [4, 5]. An advantage of the former method is the preservation of the canonical formalism and the linearity of the theory as well as the fact that its operators can be derived directly from the system-plus-reservoir approach [6].

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The physical equivalence between this approach and one of the above-mentioned nonlinear Schrödinger equations (with logarithmic nonlinearity) has been shown in [7]. Since in this paper we want to study the influence of dissipative frictional forces on the classical as well as on the quantum-mechanical level, we concentrate here on the canonical formalism using time-dependent Hamiltonians.

A problem of this type is the motion of a charged particle in a constant magnetic field when immersed in a medium that exerts some friction. We shall start with the classical analysis of the problem without dissipation and then introduce the friction term. That the dissipative magnetic field problem is already non-trivial on the classical level can be seen, e.g., from the discussions in [8]. The situation becomes even more problematic on the quantum-mechanical level when, particularly, the above-mentioned time-dependent Hamiltonian is used (see e.g., Wagner's criticism [9] of such approaches [10]). The crucial point is that in these cases the friction contribution in the corresponding equations of motion contains a term that explicitly depends on the vector potential  $\mathbf{A}$ , thus, causing rejection of this method [9] on physical grounds. We will show how this serious shortcoming can be eliminated and a consistent description of the dissipative case can be reached, also when a magnetic field is present.

Once we have our classical results we turn to the quantum problem, first without friction, but analysing it in terms of coherent states so that we have a solution corresponding to the classical one, and then we see, from the friction effect in the classical problem, what we can expect for the quantum-mechanical one.

Finally, we solve the dynamical quantum problem, without and with dissipation, by constructing time-dependent Gaussian wave packets as a superposition of the coherent states showing that their behaviour agrees with the aforementioned classical and quantum-mechanical results.

## 2. The classical problem without dissipation

The Lagrangian for the problem of a particle in a constant magnetic field is [11]

$$L = \frac{m'}{2}v'^2 + \frac{e}{c}\mathbf{v}' \cdot \mathbf{A}' - V(\mathbf{r}) \quad (1)$$

where

$$\mathbf{A}' = \frac{1}{2}(\mathbf{B}' \times \mathbf{r}') \quad (2)$$

and we indicate all observables in cgs units by primes, so as to reserve the unprimed letters for the corresponding observables in atomic units which will be used throughout this paper. Bold-face quantities denote vectors. Thus  $\mathbf{v}'$  stands for the velocity of the particle and  $\mathbf{B}'$  for the intensity of the magnetic field, both in terms of cgs units.

The relation between cgs and atomic units for all relevant observables is given by the following equations

$$\mathbf{r} = \frac{me^2}{\hbar^2}\mathbf{r}' \quad \mathbf{p} = \frac{\hbar}{me^2}\mathbf{p}' \quad \mathbf{B} = \frac{\hbar^4}{m^2e^5}\mathbf{B}' \quad (3)$$

$$\mathbf{v} = \frac{\hbar}{e^2}\mathbf{v}' \quad t = \frac{me^4}{\hbar^3}t' \quad E = \frac{\hbar^2}{me^4}E' \quad (4)$$

where  $m$ ,  $e$  are respectively the mass and charge of the particle. If the particle is an electron, relations (3) and (4) give the usual definition of atomic units [12] but then can be extended to any particle if we use units in which  $m = e = \hbar = 1$ .

The Lagrangian of equation (1), in the energy units indicated in equation (4), then becomes

$$L = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + b(x_1\dot{x}_2 - x_2\dot{x}_1) - V(\mathbf{r}) \quad (5)$$

where we assumed that the direction of the magnetic field is that of  $x_3$  and

$$b = \frac{1}{2} \frac{\hbar^3 B'}{m^2 e^3 c} \quad (6)$$

while the dots indicate derivations with respect to the time  $t$ . The motion in the  $x_3$  direction will be that of a free particle so we have suppressed the term  $(\dot{x}_3^2/2)$  in the Lagrangian of equation (5).

The Euler–Lagrange equations of motion [13] are then given by

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \quad i = 1, 2 \quad (7)$$

and for the  $L$  of equation (5), without external potential  $V(\mathbf{r})$ , they become

$$\ddot{x}_1 - 2b\dot{x}_2 = 0 \quad \ddot{x}_2 + 2b\dot{x}_1 = 0. \quad (8)$$

We solve them by replacing the Cartesian  $x_1, x_2$  by the circular ones  $x_{\pm}$

$$x_{\pm} \equiv \frac{1}{\sqrt{2}}(x_1 \pm ix_2) \quad (9)$$

for which equations (8) take the form

$$\frac{d}{dt}(\dot{x}_{\pm} \pm i2bx_{\pm}) = 0. \quad (10)$$

This means that the expression inside the brackets is a time-independent constant which we denote by  $\pm i2bx_{0\pm}$  and so we get

$$\dot{x}_{\pm} \pm i(2bx_{\pm}) = \pm i2bx_{0\pm} \quad (11)$$

which is an inhomogeneous first-order linear equation whose solution is given by

$$x_{\pm} = A_{\pm} \exp[-(\pm 2ib)t] + x_{0\pm} \quad (12)$$

where  $A_{\pm}$  is a time-independent constant. As  $x_{\pm} = x_{\mp}^*$ , where the star stands for complex conjugation, we require that  $A_{\pm} = A_{\mp}^*$ , but otherwise it is an arbitrary complex number.

The difference  $x_{\pm} - x_{0\pm}$  can now be expressed in terms of polar coordinates by the definition

$$x_{\pm} - x_{0\pm} = r \exp(\pm i\varphi) \quad (13)$$

which from equation (12) leads to the relation

$$\frac{1}{\sqrt{2}} r e^{\pm i\varphi} = A_{\pm} \exp[(\mp 2ib)t] \quad (14)$$

and implies that

$$\frac{1}{2} r^2 = A_+ A_- \equiv r_0^2 \quad \varphi = -2bt \quad (15)$$

so the motion is in a circle of radius  $r_0$  with an angular velocity  $\dot{\varphi} = -2b$ .

The velocity of the particle in its circular orbit is  $r_0\dot{\varphi} = (-2b)r_0$  so that the energy, in our units where the mass is 1, becomes

$$\frac{1}{2}(r_0\dot{\varphi})^2 = 2b^2 r_0^2 \equiv E_0. \quad (16)$$

The motion of our particle in the  $(x_1, x_2)$  plane is given by circles whose centre point  $x_{0\pm}$  is arbitrary but the radius squared is

$$r_0^2 = E_0/2b^2 \quad (17)$$

with  $E_0$  being the constant energy of the particle and  $2b$  the absolute value of the angular velocity.

### 3. The classical problem with dissipation

The simplest way to introduce dissipation is by adding to the Newtonian-type equations of motion (8) a term that is proportional to the velocity. But we have shown before [1] that the desired equations of motion (without magnetic field) can also be obtained along the lines of a modified Hamiltonian formalism, also having the advantage that the Hamiltonian function allows canonical quantization. In the following, a brief outline of this method will be given and the modifications, due to the presence of a magnetic field, will be specified.

Dissipative systems with linear velocity dependent frictional forces cannot be described in the usual Hamiltonian formalism in terms of the physical position variables  $x_i$  and momentum variables  $p_i$  of the system alone (without environmental degrees of freedom). However, it is possible to change from the physical variables to a set of so-called canonical variables, denoted by  $\bar{x}_i$  and  $\bar{p}_i$ , via a non-canonical transformation. For these variables, a Hamiltonian exists and the usual formalism is applicable. The physical results are finally obtained via the inverse transformation. In particular, on the canonical level, the Hamilton–Jacobi equation

$$\frac{\partial \bar{S}}{\partial t} + \bar{H} \left( \bar{x}_i, \frac{\partial \bar{S}}{\partial \bar{x}_i}, t \right) = 0 \quad (18)$$

for the transformed action  $\bar{S}$  and Hamiltonian  $\bar{H}$  is valid. They are connected with the physical action  $S$  and the Hamiltonian  $H$  of the corresponding conservative system via [1]

$$\bar{S} = S e^{\gamma t} \quad \bar{H} = H e^{\gamma t} \quad (19)$$

where  $\gamma$  is a constant, the so-called friction coefficient. From the definition of  $\bar{S}$ , the canonical momentum  $\bar{p}_i$  follows immediately, as

$$\bar{p}_i = \frac{\partial \bar{S}}{\partial \bar{x}_i} = e^{\gamma t} \frac{\partial S}{\partial x_i} = e^{\gamma t} p_i \quad (20)$$

where the position variable remains unchanged, i.e.  $\bar{x}_i = x_i$ . Therefore, the transformation from the physical variables  $x_i, p_i$  to the canonical variables  $\bar{x}_i, \bar{p}_i$  is non-canonical. The Hamiltonian  $\bar{H}$  can be expressed in terms of  $\bar{p}_i$  and attains a form that was proposed by Caldirola and Kanai [3] for the description of dissipative systems.

To obtain  $\bar{H}$  for the motion in a constant magnetic field, we first need the Hamiltonian  $H$  of the corresponding conservative problem. In our case, we start from the Lagrangian of equation (5), knowing that the variables canonically conjugate to the coordinates are the momenta given by

$$\frac{\partial L}{\partial \dot{x}_1} = p_1 = \dot{x}_1 - bx_2 = \dot{x}_1 + A_1 \quad \frac{\partial L}{\partial \dot{x}_2} = p_2 = \dot{x}_2 + bx_1 = \dot{x}_2 + A_2 \quad (21)$$

from which we have

$$\dot{x}_1 = p_1 + bx_2 = p_1 - A_1 \quad \dot{x}_2 = p_2 - bx_1 = p_2 - A_2. \quad (22)$$

The Hamiltonian is then given by

$$H = (\dot{x}_1 p_1 + \dot{x}_2 p_2) - L(x_i, \dot{x}_i) \quad (23)$$

where  $L$  is the Lagrangian of equation (5) and the  $\dot{x}_i$  are replaced by the  $p_i$  through equation (22). Carrying out the replacement indicated, we obtain

$$H = \frac{1}{2} (\mathbf{p} - \mathbf{A})^2 + V(\mathbf{r}). \quad (24)$$

Using our definitions from above, the dissipative Hamiltonian  $\bar{H}$  can be written as

$$\bar{H} = \frac{1}{2} e^{-\gamma t} (\bar{\mathbf{p}} - e^{\gamma t} \mathbf{A})^2 + e^{\gamma t} V(\mathbf{r}) = \bar{T} + \bar{V}. \quad (25)$$

For reasons that will become obvious later, we will keep the potential term  $\bar{V}$  on the canonical level—even if in the conservative case no potential  $V(\mathbf{r})$  might be present.

So far, the Hamiltonians considered only describe the mechanical aspect of the system but not yet the contributions from the electric and magnetic fields. These can be taken into account by

$$H_{\text{field}} = \int d\mathbf{r} \mathcal{H}_{\text{field}} \quad (26)$$

with the Hamiltonian density

$$\mathcal{H}_{\text{field}} = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \quad (27)$$

where  $\mathbf{E}$  is the electric field vector and the total Hamiltonian is given by

$$H_{\text{tot}} = H + H_{\text{field}}. \quad (28)$$

From the Lagrangian corresponding to the field contribution,  $L_{\text{field}} = \int d\mathbf{r} \mathcal{L}_{\text{field}}$  with  $\mathcal{L}_{\text{field}} = \frac{1}{8\pi} (\mathbf{E}^2 - \mathbf{B}^2)$ , Maxwell's equations can be derived [14], where the homogeneous ones are equivalent to

$$\frac{\partial}{\partial t} \mathbf{A} = -\mathbf{E} - \nabla\Phi \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (29)$$

with the nabla operator  $\nabla$  and the scalar electric potential  $\Phi$ .

Transition to the canonical level, including dissipation, should also change, according to (19),  $\mathcal{H}_{\text{field}}$  into  $\bar{\mathcal{H}}_{\text{field}}$  according to

$$\bar{\mathcal{H}}_{\text{field}} = e^{\gamma t} \mathcal{H}_{\text{field}}. \quad (30)$$

On the canonical level, the form of the equations (29) should be unchanged, i.e.

$$\frac{\partial}{\partial t} \bar{\mathbf{A}} = -\bar{\mathbf{E}} - \nabla\bar{\Phi} \quad \bar{\mathbf{B}} = \nabla \times \bar{\mathbf{A}} \quad (31)$$

should be valid. Since, apart from the electric charge  $e$  (that is 1 in our units),  $\Phi$  can be identified with the potential  $V$  if no external potentials are present, it follows from (25) that

$$\bar{\Phi} = \bar{V} = e^{\gamma t} V = e^{\gamma t} \Phi. \quad (32)$$

Inserting this into the first equation (31), a way to keep equations (31) consistent is, therefore, to also define  $\bar{\mathbf{E}} = e^{\gamma t} \mathbf{E}$ ,  $\bar{\mathbf{B}} = e^{\gamma t} \mathbf{B}$  and  $\bar{\mathbf{A}} = e^{\gamma t} \mathbf{A}$ . This would imply that  $\bar{\mathcal{L}}_{\text{field}} = \frac{1}{8\pi} e^{\gamma t} (\mathbf{E}^2 - \mathbf{B}^2) = e^{\gamma t} \mathcal{L}_{\text{field}}$ , which is also in agreement with the Caldirola–Kanai form [3] for the dissipative Lagrangian of a mechanical system.

$\bar{\mathcal{H}}_{\text{field}}$  can then be written in a form like  $\bar{T}$ , namely

$$\bar{\mathcal{H}}_{\text{field}} = \frac{1}{8\pi} e^{-\gamma t} (\bar{\mathbf{E}}^2 + \bar{\mathbf{B}}^2) \quad (33)$$

and the mechanical Hamiltonian takes the form

$$\bar{H} = \frac{1}{2} e^{-\gamma t} (\bar{\mathbf{p}} - \bar{\mathbf{A}})^2 + e\bar{\Phi}. \quad (34)$$

From this Hamiltonian (and with  $\mathbf{A} = b(-x_2, x_1)$ ) the canonical equations of motion follow as

$$\dot{\bar{x}}_1 = \dot{x}_1 = \frac{\partial \bar{H}}{\partial \bar{p}_1} = e^{-\gamma t} (\bar{p}_1 - \bar{A}_1) = p_1 - A_1 = p_1 + bx_2 \quad (35)$$

or

$$\bar{p}_1 = e^{\gamma t} (\dot{x}_1 + A_1) = e^{\gamma t} (\dot{x}_1 - bx_2) \quad (36)$$

$$\dot{\bar{x}}_2 = \dot{x}_2 = \frac{\partial \bar{H}}{\partial \bar{p}_2} = e^{-\gamma t} (\bar{p}_2 - \bar{A}_2) = p_2 - A_2 = p_2 - bx_1 \quad (37)$$

or

$$\bar{p}_2 = e^{\gamma t} (\dot{x}_2 + A_2) = e^{\gamma t} (\dot{x}_2 + bx_1) \quad (38)$$

and

$$\dot{\bar{p}}_1 = -\frac{\partial \bar{H}}{\partial \bar{x}_1} = (p_2 - A_2) \frac{\partial}{\partial x_1} (\bar{A}_2) - \frac{\partial}{\partial x_1} \bar{\Phi} = e^{\gamma t} \left( b\dot{x}_2 - \frac{\partial}{\partial x_1} \Phi \right) \quad (39)$$

$$\dot{\bar{p}}_2 = -\frac{\partial \bar{H}}{\partial \bar{x}_2} = (p_1 - A_1) \frac{\partial}{\partial x_2} (\bar{A}_1) - \frac{\partial}{\partial x_2} \bar{\Phi} = e^{\gamma t} \left( -b\dot{x}_1 - \frac{\partial}{\partial x_2} \Phi \right). \quad (40)$$

Comparing (39) and (40) with the time derivatives of (36) and (38), respectively, one finds

$$\ddot{x}_1 - 2b\dot{x}_2 + \gamma\dot{x}_1 + \gamma A_1 + \frac{\partial}{\partial x_1} \Phi = 0 \quad \ddot{x}_2 + 2b\dot{x}_1 + \gamma\dot{x}_2 + \gamma A_2 + \frac{\partial}{\partial x_2} \Phi = 0 \quad (41)$$

i.e., equations of motion which contain the above-mentioned unphysical terms proportional to the vector potential  $\mathbf{A}$ . However, the transformation of the Hamiltonian field  $\mathcal{H}_{\text{field}}$  according to (30), with the corresponding equations (31), makes it possible to eliminate this problem.

From the first equation (31), it follows (in the absence of an electric field  $\mathbf{E}$ ) that

$$\nabla \bar{\Phi} = e^{\gamma t} \nabla \Phi = -\frac{\partial}{\partial t} \bar{\mathbf{A}} = -\frac{\partial}{\partial t} (e^{\gamma t} \mathbf{A}). \quad (42)$$

For a time-independent vector potential  $\mathbf{A}$  on the physical level, i.e.  $\frac{\partial}{\partial t} \mathbf{A} = 0$ , it follows with

$$\frac{\partial}{\partial t} \bar{\mathbf{A}} = \gamma \bar{\mathbf{A}} = \gamma e^{\gamma t} \mathbf{A} \quad (43)$$

that

$$\frac{\partial}{\partial x_i} \bar{\Phi} = -\gamma A_i \quad (44)$$

and, thus, the last two terms on the lhs of equations (41) cancel.

Note that without dissipation, i.e.  $\gamma = 0$ , the term  $\nabla \bar{\Phi}$  and, consequently,  $\nabla \Phi$  disappears, i.e.  $\Phi$  and, thus,  $V$  on the physical level is just a constant (with respect to spatial variables) and will be chosen to be zero in the following.

So, finally, we obtain the desired equations of motion

$$\ddot{x}_1 - 2b\dot{x}_2 + \gamma\dot{x}_1 = 0 \quad \ddot{x}_2 + 2b\dot{x}_1 + \gamma\dot{x}_2 = 0 \quad (45)$$

where the friction constant  $\gamma$  is a measure of the strength of the dissipative term which, besides, is isotropic as it has the same value for the directions  $x_1$  and  $x_2$ .

We solve equations (45) by the same procedure that we used for the non-dissipative problem, i.e. by introducing the circular components  $x_{\pm}$  defined in equation (9), that now satisfy the equation

$$\frac{d}{dt} (\dot{x}_{\pm} \pm i2bx_{\pm} + \gamma x_{\pm}) = 0 \quad (46)$$

whose solution is given by

$$x_{\pm} = x_{0\pm} + A_{\pm} \exp[-(\pm i2b + \gamma)t] \quad (47)$$

where, as in the discussion following equation (12),  $A_{\pm} = A_{\mp}^*$ .

Again, as in equation (13), we can express the differences  $x_{\pm} - x_{0\pm}$  in terms of polar coordinates and note that from  $A_{\pm} = A_{\mp}^*$  we can write them as

$$A_{\pm} = \frac{1}{\sqrt{2}} r_0 e^{\pm i\delta} \quad (48)$$

with  $r_0$  and  $\delta$  real constants, so we now get

$$\frac{1}{\sqrt{2}} r \exp(\pm i\varphi) = \frac{1}{\sqrt{2}} [r_0 \exp(-\gamma t)] \exp[\pm i(-2bt + \delta)]. \quad (49)$$

From equation (49) it follows that

$$r^2 = r_0^2 \exp(-2\gamma t) \quad (50)$$

$$\varphi = -2bt + \delta \quad (51)$$

so the angular velocity  $\dot{\varphi} = -2b$  remains the same as for the non-dissipative problem but now the square of the radius is time dependent. From equation (51) we can express the time  $t$  in terms of  $\varphi$  and thus get for the square of the radius the expression

$$r^2 = r_0^2 \exp[(\gamma/b)(\varphi - \delta)] \quad (52)$$

which is an equation for a spiral.

We wish now to discuss the energy in this problem which we can do starting from the equations of motion in equation (45).

We multiply the first of the latter by  $\dot{x}_1$  and the second by  $\dot{x}_2$  and add them to obtain

$$\frac{d}{dt} \left[ \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) \right] + \gamma (\dot{x}_1^2 + \dot{x}_2^2) = 0. \quad (53)$$

The energy is then given by

$$E = \frac{1}{2} (\dot{x}_1^2 + \dot{x}_2^2) \quad (54)$$

and satisfies the equation

$$\frac{dE}{dt} + 2\gamma E = 0 \quad (55)$$

from which it follows that

$$E = E_0 \exp(-2\gamma t). \quad (56)$$

Comparing equations (56) and (50) we see that they have the same behaviour as functions of time and should be related. In fact we can find this relation by noting that from equation (9)

$$\dot{x}_1^2 + \dot{x}_2^2 = 2\dot{x}_+ \dot{x}_- \quad (57)$$

and from equation (49)

$$\dot{x}_{\pm} = \frac{1}{\sqrt{2}} (\mp 2bi - \gamma) r_0 \exp(-\gamma t) \exp[\pm i(-2bt + \delta)] \quad (58)$$

so the energy is now given by

$$E = \frac{1}{2} (4b^2 + \gamma^2) r_0^2 \exp(-2\gamma t) \quad (59)$$

and comparing with equation (56) we obtain

$$E_0 = \left( 2b^2 + \frac{\gamma^2}{2} \right) r_0^2. \quad (60)$$

The energy, in the presence of dissipative forces, is not conserved and in fact it decays with time. If the dissipative effect is small compared to the cyclotron energy, i.e. if

$$\gamma^2 \ll 4b^2 \quad (61)$$

then we have as in equation (17) the relation obtained previously,  $r_0^2 = E_0/2b^2$ .



#### 4. The quantum-mechanical problem without dissipation

To obtain the quantum-mechanical equations, we take the Hamiltonian (24) (with  $V = 0$ , as mentioned above), using the definition (2) of  $\mathbf{A}$  (in our units) and writing it in components as

$$H = \frac{1}{2} [(p_1^2 + p_2^2) + b^2(x_1^2 + x_2^2)] + b[-(x_1 p_2 - x_2 p_1)]. \quad (62)$$

The corresponding operator is obtained by usual canonical quantization, substituting  $p_i$  by  $-i\frac{\partial}{\partial x_i}$ .

We would like to solve this problem in such a way that the eigenstates are related in a simple manner to the behaviour we observed in the classical discussion in section 2.

For this purpose we express the Hamiltonian operator in terms of the creation and annihilation operators

$$\eta_i = \frac{1}{\sqrt{2}} \left( \sqrt{b}x_i - \frac{i}{\sqrt{b}}p_i \right) \quad \xi_i = \frac{1}{\sqrt{2}} \left( \sqrt{b}x_i + \frac{i}{\sqrt{b}}p_i \right) \quad (63)$$

and find that the Hamiltonian can be expressed as

$$H = b[(\eta_1\xi_1 + \eta_2\xi_2 + 1) - (1/i)(\eta_1\xi_2 - \eta_2\xi_1)]. \quad (64)$$

We then pass to what we could call the circular creation and annihilation operators defined by

$$\eta_{\pm} = \frac{1}{\sqrt{2}}(\eta_1 \pm i\eta_2) \quad \xi_{\pm} = \frac{1}{\sqrt{2}}(\xi_1 \mp i\xi_2) \quad (65)$$

and in terms of them the Hamiltonian takes the form

$$H = b[(\eta_+\xi_+ + \eta_-\xi_- + 1) - (\eta_+\xi_+ - \eta_-\xi_-)] = b(2\eta_-\xi_- + 1). \quad (66)$$

The creation and annihilation operators of equation (63) clearly satisfy the commutation relations

$$[\eta_i, \eta_j] = [\xi_i, \xi_j] = 0 \quad [\xi_i, \eta_j] = \delta_{ij} \quad i, j = 1, 2 \quad (67)$$

and similar relations hold for  $\eta_{\pm}, \xi_{\pm}$ .

Using now the circular coordinates

$$x_{\pm} = \frac{1}{\sqrt{2}}(x_1 \pm ix_2) \quad (68)$$

we conclude from equations (63), (65) and (68) that  $\eta_{\pm}, \xi_{\pm}$  have the differential form

$$\eta_{\pm} = \frac{1}{\sqrt{2}} \left( \sqrt{b}x_{\pm} - \frac{1}{\sqrt{b}}\frac{\partial}{\partial x_{\mp}} \right) \quad \xi_{\pm} = \frac{1}{\sqrt{2}} \left( \sqrt{b}x_{\mp} + \frac{1}{\sqrt{b}}\frac{\partial}{\partial x_{\pm}} \right). \quad (69)$$

The Hamiltonian  $b(2\eta_-\xi_- + 1)$  of equation (66) does not contain  $\eta_+, \xi_+$ , so we are completely free in the ways we can characterize the eigenstates of  $H$  in relation to these last operators. In this section we shall consider eigenstates of both  $(2\eta_-\xi_- + 1)$  and  $\xi_+$ . Since the latter is a non-Hermitian operator [15] we shall designate its eigenvalue by the complex number  $z_0$  whose real and imaginary parts, for later convenience, we denote by  $\sqrt{b}x_{01}$  and  $\sqrt{b}x_{02}$  respectively, i.e.

$$z_0 = \sqrt{b}(x_{01} + ix_{02}). \quad (70)$$

Since  $[\xi_-, \eta_-] = 1$ ,  $\eta_-\xi_-$  is a number operator and we shall designate its eigenvalues by  $\nu = 0, 1, 2, \dots$ . We are thus looking for kets  $|\nu, z_0\rangle$  satisfying

$$(2\eta_-\xi_- + 1)|\nu, z_0\rangle = (2\nu + 1)|\nu, z_0\rangle \quad (71)$$

$$\xi_+|\nu, z_0\rangle = z_0|\nu, z_0\rangle \quad (72)$$

which we shall call coherent states [16] since those are precisely characterized by equation (72).

To obtain  $|\nu, z_0\rangle$  we start [17]<sup>3</sup> with a ground state  $|0\rangle$  obeying

$$\xi_{\pm}|0\rangle = 0 \tag{73}$$

which from equation (69) takes the form

$$|0\rangle = \left(\frac{b}{\pi}\right)^{1/2} \exp[-bx_+x_-] = \left(\frac{b}{\pi}\right)^{1/2} \exp[-(b/2)(x_1^2 + x_2^2)]. \tag{74}$$

From the commutation relations (67), which hold even if  $i, j = +, -$ , we see that to obtain the eigenvalue  $\nu$  for  $\eta_- \xi_-$  we need to apply  $\eta_-^\nu$  to  $|0\rangle$ , while to obtain the eigenvalue  $z_0$  for  $\xi_+$  we need to apply  $\exp(z_0\eta_+)$  to  $|0\rangle$ . Thus the normalized coherent state  $|\nu z_0\rangle$  is given by [17]

$$|\nu, z_0\rangle = \left[(\nu!)^{-\frac{1}{2}} \eta_-^\nu\right] \exp(-1/2|z_0|^2) \exp(z_0\eta_+) |0\rangle \tag{75}$$

where  $|z_0|^2 = b(x_{01}^2 + x_{02}^2)$  and  $\exp(z_0\eta_+)$  should be understood as the exponential series in powers of  $z_0\eta_+$ .

From equations (69) and (74) we see that

$$\eta_+|0\rangle = \sqrt{2b}x_+|0\rangle \tag{76}$$

and thus in equation (75) we can replace  $\eta_+$  by  $\sqrt{2b}x_+$  to obtain

$$|\nu, z_0\rangle = \sqrt{b}(\pi\nu!)^{-\frac{1}{2}} \exp[-\frac{1}{2}b(x_{01}^2 + x_{02}^2)] \eta_-^\nu \exp[(\sqrt{2}z_0 - \sqrt{b}x_-)\sqrt{b}x_+]. \tag{77}$$

Finally, applying  $\nu$  times  $\eta_-$  we obtain [17]

$$|\nu, z_0\rangle = b^{\frac{1}{2}}(\pi\nu!)^{-\frac{1}{2}} b^{\frac{\nu}{2}} [(x_1 - x_{01}) + i(x_2 - x_{02})]^\nu \times \exp\{-(b/2)[(x_1 - x_{01})^2 + (x_2 - x_{02})^2]\} \exp[ib(x_{02}x_1 - x_{01}x_2)]. \tag{78}$$

This state is normalized and besides we obtain [17]

$$\langle\nu, z_0|x_i|\nu, z_0\rangle = x_{0i} \quad i = 1, 2 \tag{79}$$

$$\langle\nu, z_0|(x_1 - x_{01})^2 + (x_2 - x_{02})^2|\nu, z_0\rangle = b^{-1}(\nu + 1) = [b(\nu + 1)/b^2]. \tag{80}$$

Relation (79) is obtained by noting that  $x_i - x_{0i}$  changes sign when we make a reflection  $x_i - x_{0i} \rightarrow -(x_i - x_{0i})$  while all the other terms in equation (79) are invariant under reflection. Thus the expectation value of  $x_i - x_{0i}$  with respect to the states  $|\nu, z_0\rangle$  is 0 from which it follows that the expectation value of  $x_i$  is  $x_{0i}$ . Thus the coherent eigenstate  $|\nu, z_0\rangle$  of the Hamiltonian  $H$  is centred at the point  $(x_{01}, x_{02})$ .

To understand the result in equation (80) we note that the probability density for states  $|\nu, z_0\rangle$  is given by

$$|\langle x_1, x_2|\nu, z_0\rangle|^2 = b(\pi\nu!)^{-1} \rho^{2\nu} \exp(-\rho^2) \tag{81}$$

where

$$\rho^2 \equiv b[(x_1 - x_{01})^2 + (x_2 - x_{02})^2] \tag{82}$$

and it is normalized as

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\langle x_1 x_2|\nu z_0\rangle|^2 dx_1 dx_2 = \frac{1}{\nu!} \int_0^{\infty} (\rho^2)^\nu \exp(-\rho^2) d\rho^2 = 1 \tag{83}$$

where the element  $dx_1 dx_2$  in polar coordinates becomes  $b^{-1} \rho d\rho d\varphi$  which cancels the factor  $b$  in equation (81), and where an integration over  $\varphi$  gives  $2\pi$ .

<sup>3</sup> Note that in this reference  $b$  is taken as negative so that  $\eta_+$  replaces the operator  $\eta_-$  of the present paper but all the conclusions remain valid.

Taking into account that the expectation value of  $\rho^2$  changes only  $(\rho^2)^\nu$  to  $(\rho^2)^{\nu+1}$  in equation (83) we obtain  $(\nu + 1)$  for its value and from equation (82) this justifies the result of equation (80).

We note from equation (66) that the quantum-mechanical energy, which for a non-dissipative problem we shall denote by  $E_0$ , is given by

$$E_0 = b(2\nu + 1) \quad (84)$$

and thus when  $\nu \gg 1$ , equation (80) agrees with the classical value of the square of the radius given by  $E_0/2b^2$  in equation (17).

To understand more fully what the coherent state wavefunction  $|\nu, z_0\rangle$  implies we note from equation (81) that the maximum of the probability density occurs at  $\rho^2 = \nu$  and there it takes the value

$$|\langle x_1, x_2 | \nu, z_0 \rangle|_{\max}^2 = b(\pi \nu!)^{-1} \nu^\nu e^{-\nu} \simeq b\pi^{-1} (2\pi \nu)^{-\frac{1}{2}} \quad (85)$$

in which the right-hand side is valid for large  $\nu$  when we use Stirling's formula.

In the classical limit, i.e. when  $\nu \gg 1$ , we have from equation (80) that the probability density is concentrated in a ring of radius  $(\nu/b)^{\frac{1}{2}}$  and width  $\Delta$  where the latter can be estimated by considering that the product of  $(2\pi(\nu/b)^{\frac{1}{2}}\Delta)$  with the maximum height of equation (85) of the probability density should give 1, since the wavefunction  $|\nu, z_0\rangle$  is normalized. For  $\nu \gg 1$  we thus have

$$\left(2\pi \nu^{\frac{1}{2}} b^{-\frac{1}{2}} \Delta\right) \left(\pi^{-1} (2\pi \nu)^{-\frac{1}{2}} b\right) = (2/\pi)^{\frac{1}{2}} b^{\frac{1}{2}} \Delta \simeq 1. \quad (86)$$

Thus the value of  $\Delta = (\pi/2)^{\frac{1}{2}} b^{-\frac{1}{2}}$  and becomes very small for large values of the cyclotron frequency in atomic units.

Thus the coherent wavefunctions represent the states associated with the classical circular orbits of a definite energy in a constant magnetic field. Besides the relation between the square of the radius of the orbit and the ratio of half the energy with the square of the angular velocity is still valid in this quantum system as seen in equation (80).

## 5. The quantum-mechanical problem with dissipation

For comparison with the results of the previous section, we write the canonical Hamiltonian  $\bar{H}$ , also in components, in terms of  $\bar{x}_i = x_i$  and  $\bar{p}_i$  (without considering the potential  $\bar{V}$  that is related to the explicit time-dependence of the vector potential  $\bar{\mathbf{A}}$ , this has to be taken into account in the time-dependent treatment of the problem).

The classical Hamiltonian

$$\bar{H} = \frac{1}{2} [e^{-\gamma t} (\bar{p}_1^2 + \bar{p}_2^2) + b^2 e^{\gamma t} (x_1^2 + x_2^2)] + b[-(x_1 \bar{p}_2 - x_2 \bar{p}_1)] \quad (87)$$

can be quantized straightforwardly, now replacing the canonical momentum  $\bar{p}_i$  by  $-i\frac{\partial}{\partial x_i}$ .

Again, the Hamiltonian can be expressed in terms of operators such as the creation and annihilation operators  $\eta_i$  and  $\xi_i$ , now defined as

$$\begin{aligned} \bar{\eta}_i &= \frac{1}{\sqrt{2}} \left( \sqrt{b} e^{\gamma t/2} x_i - \frac{i}{\sqrt{b}} e^{-\gamma t/2} \bar{p}_i \right) = \frac{1}{\sqrt{2}} \left( \sqrt{b} e^{\gamma t/2} x_i - \frac{1}{\sqrt{b}} e^{-\gamma t/2} \frac{\partial}{\partial x_i} \right) \\ \bar{\xi}_i &= \frac{1}{\sqrt{2}} \left( \sqrt{b} e^{\gamma t/2} x_i + \frac{i}{\sqrt{b}} e^{-\gamma t/2} \bar{p}_i \right) = \frac{1}{\sqrt{2}} \left( \sqrt{b} e^{\gamma t/2} x_i + \frac{1}{\sqrt{b}} e^{-\gamma t/2} \frac{\partial}{\partial x_i} \right) \end{aligned} \quad (88)$$

yielding

$$\bar{H} = b[(\bar{\eta}_1 \bar{\xi}_1 + \bar{\eta}_2 \bar{\xi}_2 + 1) - (1/i)(\bar{\eta}_1 \bar{\xi}_2 - \bar{\eta}_2 \bar{\xi}_1)]. \quad (89)$$

Changing again to the corresponding circular operators, defined in analogy with  $\eta_{\pm}$  and  $\xi_{\pm}$ , finally leads to

$$\bar{H} = b[(\bar{\eta}_+\bar{\xi}_+ + \bar{\eta}_-\bar{\xi}_- + 1) - (\bar{\eta}_+\bar{\xi}_+ - \bar{\eta}_-\bar{\xi}_-)] = b(2\bar{\eta}_-\bar{\xi}_- + 1) \quad (90)$$

i.e. the same form as in equation (66), only with barred quantities. The ground state  $|\bar{0}\rangle$  can now be obtained from

$$\bar{\xi}_{\pm}|\bar{0}\rangle = \frac{1}{\sqrt{2}} \left( \sqrt{b} e^{\gamma t/2} x_{\mp} + \frac{1}{\sqrt{b}} e^{-\gamma t/2} \frac{\partial}{\partial x_{\pm}} \right) |\bar{0}\rangle = 0 \quad (91)$$

and takes the normalized form

$$|\bar{0}\rangle = \left( \frac{b e^{\gamma t}}{\pi} \right)^{1/2} \exp[-b e^{\gamma t} x_+ x_-] = \left( \frac{b e^{\gamma t}}{\pi} \right)^{1/2} \exp[-(b/2) e^{\gamma t} (x_1^2 + x_2^2)]. \quad (92)$$

Due to the formal similarity between  $\bar{\eta}_i, \bar{\xi}_i$  and the corresponding quantities without bar, the construction of coherent states with higher  $\nu$  should follow the same scheme as in the case without dissipation, except that now an exponential factor occurs, in particular in the exponent of the coherent states.

At this point, the relation between the action and the wavefunction, introduced by Schrödinger in his first paper on wave mechanics [18],

$$S = -i \ln |\Psi\rangle \quad (93)$$

shall be used to make the link between the coherent states on the canonical level and their dissipative counterparts on the physical level, as shown in our previous work [1].

According to equation (19), the physical and the canonical actions are related by  $S = e^{-\gamma t} \bar{S}$ , so the corresponding wavefunctions and, thus, also the coherent states are connected via the non-unitary transformation

$$\ln |\Psi\rangle = e^{-\gamma t} \ln |\bar{\Psi}\rangle. \quad (94)$$

It becomes obvious that the factor  $e^{\gamma t}$  in the exponent of the canonical coherent states is compensated by the factor  $e^{-\gamma t}$  on the rhs of (94). From this one can conclude that on the physical level, also in the dissipative case, the maximum of the probability density should occur where the classical value of the radius lies.

We can, therefore, make use of our classical results to get information about the quantum problem with dissipation. We showed, classically, that the energy  $E$  for the dissipative problem is the non-dissipative one,  $E_0$ , multiplied by the decay-term  $\exp(-2\gamma t)$ . As this energy (59) is related to the mean square radius (50) by a formula that holds both classically, in equations (17) and (60), and quantum mechanically, in equation (80), we see that when the energy decays the same will happen to the mean square radius. Thus, our quantum-mechanical ring given by the probability density of equation (81) should collapse to the region around the central point, according to the decay formula  $\exp(-2\gamma t)$ .

To prove this dynamical behaviour of the maximum of the probability density we consider, in the next section, the time-dependent case (without and with dissipation) and use superpositions of the coherent states to form Gaussian wave packets that are solutions of the corresponding time-dependent Schrödinger equation.

## 6. The time-dependent quantum-mechanical problem with and without dissipation

Since the coherent states do not show the dynamical behaviour directly and also the normalization of the dissipative coherent states  $|\bar{\nu}, \bar{z}_0\rangle$  containing the  $\nu$ th power of the radius of the circular motion should not be trivial after the non-unitary transformation (94), it

seems advantageous to construct time-dependent Gaussian wave packets by a superposition of coherent states with the appropriate coefficients.

In the conservative case, these wave packets  $|\Psi(t)\rangle$  should fulfil the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle \quad (95)$$

with the Hamiltonian operator  $H$  corresponding to the Hamiltonian in equation (62). The wave packet can be expanded in terms of the coherent states  $|\nu, z_0\rangle$ ,

$$|\Psi(t)\rangle = e^{-ibt} \sum_{\nu=0}^{\infty} A_{\nu} |\nu, z_0\rangle e^{-i2\nu bt} \quad (96)$$

with time-independent coefficients  $A_{\nu}$ , in particular at  $t = 0$ :

$$|\Psi(0)\rangle = \sum_{\nu=0}^{\infty} A_{\nu} |\nu, z_0\rangle. \quad (97)$$

Assume that the initial state is a Gaussian wave packet given by

$$\begin{aligned} |\Psi(0)\rangle &= \left(\frac{b}{\pi}\right)^{1/2} \exp\left\{-\frac{b}{2}(\mathbf{r} - \langle\mathbf{r}\rangle(0))^2 + i[(\dot{\mathbf{r}}(0) + \mathbf{A}(\langle\mathbf{r}\rangle(0)))(\mathbf{r} - \langle\mathbf{r}\rangle(0))]\right\} \\ &= \left(\frac{b}{\pi}\right)^{1/2} \exp\left\{-\frac{b}{2}(\mathbf{R} - \mathbf{a}_0)^2 + i[(\dot{\mathbf{r}}(0) + \mathbf{A}(\langle\mathbf{r}\rangle(0)))(\mathbf{R} - \mathbf{a}_0)]\right\} \end{aligned} \quad (98)$$

with  $\mathbf{a}_0 = \mathbf{a}(t = 0)$ . Using the orthonormality of the  $|\nu, z_0\rangle$ , the  $A_{\nu}$  can be determined [19] and, choosing appropriate initial conditions, one gets the simple form [20]

$$A_{\nu} = \left(b^{\nu} \frac{1}{\nu!}\right)^{1/2} |\mathbf{a}_0|^{\nu} \exp\left\{-\frac{b}{2}a_0^2 - i\mathbf{A}(\mathbf{r}_0) \cdot \mathbf{a}_0\right\} \quad (99)$$

where the coefficients  $A_{\nu}$  fulfil the completeness relation  $\sum_{\nu=0}^{\infty} A_{\nu} A_{\nu}^* = 1$ . The meaning of the radial quantities  $\mathbf{R}$  and  $\mathbf{a}$  becomes obvious from figure 1.

The normalized time-dependent wave packet can be written as

$$|\Psi(t)\rangle = \left(\frac{b}{\pi}\right)^{1/2} \exp\left\{-\frac{b}{2}(\mathbf{R} - \mathbf{a}(t))^2 + i[\langle\mathbf{p}\rangle(t)(\mathbf{R} - \mathbf{a}(t)) + K(t)]\right\} \quad (100)$$

with the classical momentum  $\langle\mathbf{p}\rangle = \langle\dot{\mathbf{r}}\rangle + \mathbf{A}(\langle\mathbf{r}\rangle)$  and a purely time-dependent term  $K(t)$  that is not relevant for the following.

The corresponding density is given by

$$|\Psi(t)|^2 = \frac{b}{\pi} \exp\{-b(\mathbf{R} - \mathbf{a}(t))^2\} \quad (101)$$

i.e. the maximum follows the classical circular trajectory  $\mathbf{a}(t) = \langle\mathbf{r}\rangle(t) - \mathbf{r}_0$ .

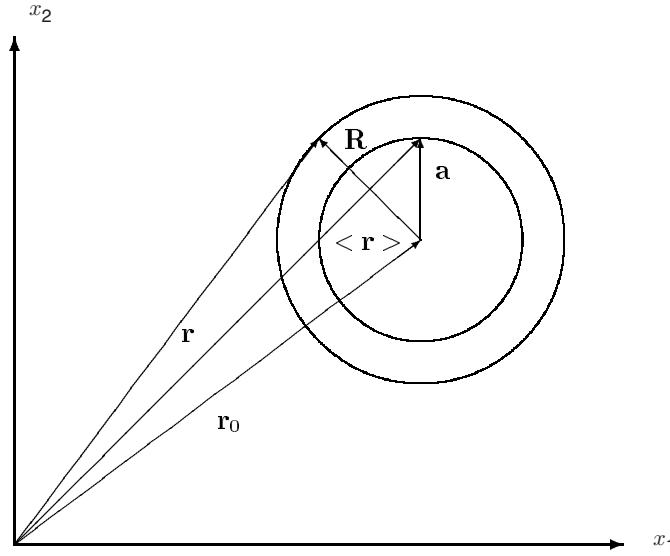
Including dissipation, a similar Gaussian wave packet can be found, fulfilling the equation

$$i\frac{\partial}{\partial t}|\bar{\Psi}(t)\rangle = \bar{H}|\bar{\Psi}(t)\rangle \quad (102)$$

with the Caldirola–Kanai Hamiltonian  $\bar{H} = \bar{T} + \bar{V}$ .

To be in agreement with the transformed classical equations for the scalar and vector potentials (see section 3), the potential  $\bar{V}$  has to fulfil

$$-\frac{\partial}{\partial x_i}\bar{V} = -e^{\gamma t}\frac{\partial}{\partial x_i}V = \gamma\bar{A}_i = \gamma e^{\gamma t}A_i. \quad (103)$$



**Figure 1.**  $\mathbf{r}$ : vector of position in the  $(x_1, x_2)$  plane;  $\langle \mathbf{r} \rangle$ : classical trajectory;  $\mathbf{r}_0$ : centre of circular motion;  $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$  with  $|\mathbf{R}|$  = radius of circular motion;  $\mathbf{a} = \langle \mathbf{r} \rangle - \mathbf{r}_0$  with  $|\mathbf{a}|$  = classical radius.

So, the potential  $\bar{V}$  on the canonical level can be given (up to a possible purely time-dependent additive term) as

$$\bar{V} = \gamma b(\langle x_2 \rangle x_1 - \langle x_1 \rangle x_2) e^{\gamma t} = -\gamma \mathbf{A}(\langle \mathbf{r} \rangle) \mathbf{r} e^{\gamma t}. \quad (104)$$

This leads to the time-dependent Gaussian wave packet

$$\begin{aligned} |\bar{\Psi}(t)\rangle &= \bar{N} \exp \left\{ -\frac{(\bar{b} + i\gamma/2)}{2} (\mathbf{R} - \mathbf{a}(t))^2 e^{\gamma t} + i[\langle \bar{\mathbf{p}} \rangle(t) (\mathbf{R} - \mathbf{a}(t)) + \bar{K}(t)] \right\} \\ &= \bar{N} \exp \left\{ e^{\gamma t} \left[ -\frac{(\bar{b} + i\gamma/2)}{2} (\mathbf{R} - \mathbf{a}(t))^2 + i[\langle \mathbf{p} \rangle(t) (\mathbf{R} - \mathbf{a}(t)) + e^{-\gamma t} \bar{K}(t)] \right] \right\} \end{aligned} \quad (105)$$

with  $\langle \bar{\mathbf{p}} \rangle = e^{\gamma t} \langle \mathbf{p} \rangle$  (where  $\langle \mathbf{p} \rangle$  is the classical dissipative momentum),  $\mathbf{a}(t)$  follows the classical dissipative trajectory, i.e. it spirals to  $\mathbf{r}_0$ ,  $\bar{b} = (b^2 - \gamma^2/4)^{1/2}$  and  $\bar{N} = (\bar{b} e^{\gamma t} / \pi)^{1/2}$  is the normalization coefficient on the canonical level.

As in the conservative case, this wave packet can also be built up by a superposition of coherent states, in this dissipative case, by the ones obtained in section 5.

The corresponding density is given by

$$|\bar{\Psi}(t)|^2 = \bar{N} \bar{N}^* \exp\{e^{\gamma t} [-\bar{b} (\mathbf{R} - \mathbf{a}(t))^2]\}. \quad (106)$$

The wave packet and density on the physical level can be found again using our relation between the actions on the different levels and between the action and the wave packet (see equations (19), (93), (94)). So, from  $\ln |\Psi\rangle = e^{-\gamma t} \ln |\bar{\Psi}\rangle$ , we get rid of the factor  $e^{\gamma t}$  in the exponent and find for the physical wave packet including dissipation

$$|\Psi_{\text{diss}}(t)\rangle \propto \exp \left\{ -\frac{(\bar{b} + i\gamma/2)}{2} (\mathbf{R} - \mathbf{a}(t))^2 + i[\langle \mathbf{p} \rangle(t) (\mathbf{R} - \mathbf{a}(t)) + e^{-\gamma t} \bar{K}(t)] \right\}. \quad (107)$$

Usual normalization of the Gaussian then yields the density

$$|\Psi_{\text{diss}}(t)|^2 = \frac{\bar{b}}{\pi} \exp\{-\bar{b} (\mathbf{R} - \mathbf{a}(t))^2\} \quad (108)$$

i.e. the maximum of the physical probability density spirals according to the classical dissipative trajectory  $\mathbf{a}(t)$  towards the origin  $\mathbf{r}_0$ , as already expected from the discussion of the dissipative coherent states at the end of section 5.

## 7. Conclusions

A classical charged particle moves in a constant magnetic field on a circle with constant radius  $r_0$ , where the radius is connected with the energy  $E_0$  via  $r_0^2 = E_0/2b^2$ . The quantum-mechanical situation can be described by coherent states, where the maximum of the corresponding probability density is concentrated in a ring of radius  $\varrho = \sqrt{\nu}$ . For large quantum numbers,  $\nu \gg 1$ , the same relation between the radius and the energy of the system is valid, as in the classical case.

If dissipation is included, the classical particle moves on a spiral with exponentially decaying radius. The energy shows the same decaying behaviour and the relation between the square of the radius and the energy is given by a constant as in the conservative case.

Quantum mechanically, a canonical description of the dissipative system is possible in terms of modified creation and annihilation operators, allowing one to obtain similar coherent states as without dissipation. A non-unitary transformation to the physical states shows that, also here, the maximum of the probability density occurs at the classical radius. Using the aforementioned relation between the radius and the energy, it can be concluded that with decaying energy, also the radius will exponentially approach the centre of the circle, i.e. states with smaller quantum numbers will be gaining higher probability.

This statement can be proved by considering the time-dependent quantum-mechanical problem. In this case, Gaussian wave packet solutions can be obtained by superimposing the corresponding coherent state solutions. In the conservative case, the maximum of the wave packet moves on a circle with the constant classical radius  $|\mathbf{a}|$ ; in the dissipative case, the maximum spirals along the classical trajectory with exponentially decaying radius  $|\mathbf{a}(t)|$  towards the centre  $\mathbf{r}_0$ .

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